1. INTRODUCTION

Complexity is often invoked alongside size and mass as a characteristic of macroscopic quantum objects. In 2004, Aaronson introduced the tree size (TS) as a computable measure of complexity and studied its basic properties. We improve and expand on those initial results. In particular, we give explicit characterizations of a family of states with superpolynomial complexity \( n^{\Omega(\log n)} = TS = O(\sqrt{n}) \) in the number of qubits \( n \).

2. MOTIVATIONS

- Testing quantum mechanics at the macroscopic scale.
- Bigger Schrödinger cats: coherent superpositions are realized with mechanical resonators, superconducting qubit, and heavy molecules.
- Complexity is an important characteristic of macroscopic systems.
- Complexity may also be relevant in the context of quantum computing.

3. TREE SIZE OF A QUANTUM STATE

- Aaronson, STOC ’04: Any quantum state can be described by a rooted tree of \( \emptyset \) and + gates. Each leaf is labeled with a single-qubit state \( |0\rangle + |1\rangle \).

- Size of a tree = number of leaves.
- Tree size of a state (TS) = size of the minimal tree = most compact way of writing the state

\[
|00\rangle + |11\rangle \quad |0\rangle + |1\rangle + (1) + (1) = \left\lfloor \frac{1 + 1}{2} \right\rfloor = \text{2 leaves}
\]

- Tree size of some well known states:

\[
TS(|00\rangle) = n \\
TS(GHZ) = 2n \\
TS(W) = O(n^2) \\
TS(1D \text{ cluster}) = O(n^2) \\
TS(2D \text{ cluster}) = 2^{\Omega(n)} \text{ conjectured} \\
TS(Shor) = \Omega(n^{\log n}) \quad \text{proved under one conjecture}
\]

- \( TS_n \leq 2^n \) for every n-qubit states (nested Schmidt decomposition).
- Upper bound on TS of a state can be obtained by finding a compact decomposition for that state\( \Rightarrow \) easy to prove that some states are NOT complex.
- Any matrix-product state whose tensors are of dimension \( D \times D \) has polynomial complexity \( TS = n^{\log D \times D} \).
- Conjectures: If a quantum state allows universal quantum computation, it must possess superpolynomial tree size, otherwise we could simulate it efficiently.

4. LINKS WITH MULTILINEAR FORMULA

- Lower bound: The tree size of a quantum state is bounded below by the multilinear formula size (MFS) of an associated multilinear formula.
- In an expansion of a state in the computational basis, the associated multilinear formula computes the coefficients

\[
|\psi\rangle = \sum_{x_1,\ldots,x_n} f(x_1,\ldots,x_n)|x_1,\ldots,x_n\rangle
\]

We want a multilinear formula to compute the coefficients

- To get the multilinear formula from the tree of a state: replace \( |0\rangle \) by \( 1 - x_1 \) and \( |1\rangle \) by \( x_1 \) by \( x \times \)

\[
\begin{cases}
\frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle\right) \\
\frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right)
\end{cases}
\]

\[
TS(|\psi\rangle) \geq \text{MFS} (f_{\psi})
\]

5. SUPERPOLYNOMIAL COMPLEX STATES (PRA 88, 012321)

- Raz, STOC ’04: any multilinear formula that computes the determinantal or permanent of a matrix is superpolynomial.
- When \( n = m^2 \), we label each computational basis by \( |x_1, x_2, \ldots, x_m\rangle \), and then arrange the variables to a matrix

\[
\{x\} = \begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1m} \\
x_{21} & x_{22} & \cdots & x_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mm}
\end{pmatrix}
\]

- These states have superpolynomial tree size:

\[
\begin{align*}
\det_m &= \sum_{x=0}^{2^n} \det(\{x\}) |x\rangle, \\
\per_m &= \sum_{x=0}^{2^n} \perm(\{x\}) |x\rangle
\end{align*}
\]

6. MOST COMPLEX FEW-QUBIT STATES

- Tree size does not change under reversible SLOCC: All states belonging to a SLOCC-equivalent family have the same tree size.
- Tree size can be found for each SLOCC-equivalent family (practical only when the number of qubits is small).

3 qubits:
- Biseparable: TS = 5
- GHZ class: TS = 6
- W class: TS = 8

Most complex, but of zero measure:
\( |W\rangle = |0\rangle + |1\rangle + |11\rangle \) is GHZ for arbitrarily small \( c \).

4 qubits:
- The most complex class can be written as \( |10\rangle |G\rangle + |10\rangle |G\rangle \) up to SLOCC \( TS = 2^4 \)
- The most complex class has finite measure.
- Example: Dicke state with two excitations \( |0011\rangle + |0101\rangle + |0110\rangle + |1010\rangle \).